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An interstellar Bussard ramjet is a spaceship using the protons of the interstellar medium in a fusion engine to produce thrust. In recent papers, it was shown that the relativistic equation of motion of an ideal ramjet and of a ramjet with radiation loss are analytical. When a mass loss appears, the limit speed of the ramjet is more strongly reduced. But, the parametric equations, in terms of the ramjet's speed, for the position of the ramjet in the inertial frame of the interstellar medium, the time in this frame, and the proper time indicated by the clocks on board the spaceship, can still be obtained in an analytical form. The non-relativistic motion and the motion near the limit speed are studied.

I. INTRODUCTION

During its motion through the space, an interstellar fusion ramjet collects interstellar ions (mostly protons) with a magnetic scoop (or ramscoop) to supply a fusion reactor able to fuse protons to obtain helium [1, 2, 3, 4]. Accelerated reaction products are exhausted out of the spacecraft's rear to produce thrust. An ideal ramjet could reach a velocity very close to the speed of light c . If some energy extracted from the interstellar medium is lost in form of thermal radiation, the ramjet speed is limited to a value below c [5]. In this two cases, we have shown that analytical formulas can be obtained for the position of the ramjet in the inertial frame of the interstellar medium, the time in this frame, and the proper time indicated by the clocks on board the spaceship [6, 7]. These parametric equations are given in terms of the ramjet's speed.

Moreover, it is natural to assume that a fraction of the collected interstellar gas can be lost during the work of the engine. The limit speed is then more strongly reduced [5] and the equation of motion further more complicated. In this paper, we show that analytical formulas can also be obtained for the position, the time and the proper time of a ramjet with radiation and mass losses. The non-relativistic motion and the motion near the limit speed are also studied.

The situation presented in this paper is more realistic than the case of a Bussard ramjet with a perfect fusion engine [6]. Nevertheless, it is not very probable that the kind of ramjet considered here could be ever built, even in the far future. Consequently, this work can be considered as an advanced exercise (from the point of view of calculus) in special relativity. The framework of interstellar space travel could be very attractive for undergraduate students. The basic equations are simple outcomes of momentum and energy conservation. Even if the solutions demand very heavy calculations, they have interesting properties to explore. However, let us note that the ramjet concept is potentially too valuable to be simply discarded despite its tremendous technical difficulties. Some researchers have suggested alternatives to the initially proposed proton-proton fusion ramjet [3, 4]: the catalytic ramjet (the use of catalyzed fusion reaction with a high rate), the RAIR (the use of nuclear fuel carried by the ship), etc.

II. GENERAL EQUATIONS

In the following, all calculations are performed in the frame of the interstellar medium, considered as an inertial frame. A Bussard ramjet of constant mass M moves at speed $v = \beta c$ through this medium, which contains protons at rest with a mass density ρ . The effective intake area of the ramscoop is denoted A . A fraction ϵ of the absorbed mass is converted into useful kinetic energy in the hydrogen fusion reactor, a fraction λ is dissipated in form of thermal

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radiation and a fraction κ is lost in the interstellar medium (parameters ϵ , λ and κ have the same meaning than in Ref. [5]). This means that, if a mass dm of protons at rest is scooped up from the interstellar medium, only an energy $\epsilon dm c^2$ is converted into ordered motion of the exhausted material.

In the frame of the spaceship, we can assume that the thermal loss is isotropic and that the radiation carries no momentum. For the ramjet, the energy absorbed due to the mass dm is $dE = dm \gamma c^2$, where $\gamma = 1/\sqrt{1-\beta^2}$, and the engine dissipates an energy equal to λdE . Lorentz transformations [8, 9] imply that an energy $\gamma \lambda dE = \lambda dm \gamma^2 c^2$ and a momentum $\gamma \beta \lambda dE/c = \lambda dm \gamma^2 \beta c$ are lost in the interstellar medium frame.

If a fraction κ of the collected interstellar gas is lost during the work of the engine, a useless mass κdm is dropped out with no velocity in the frame of the ramjet. So, this mass has an energy $\kappa dm \gamma c^2$ and a momentum $\kappa dm \gamma \beta c$ in the interstellar medium.

During the time interval dt , a mass dm of protons, at rest, is scooped-up. The ramjet speed is then increased by the quantity $cd\beta$, thanks to the ejection of a mass $(1-\alpha)dm$ of helium with a speed wc , where

$$\alpha = \epsilon + \lambda + \kappa. \quad (1)$$

Another interesting quantity is the fraction of matter,

$$\omega = \epsilon + \lambda, \quad (2)$$

which is converted into pure energy by the engine.

The conservation of momentum implies that [8, 9]

$$M \gamma(\beta) \beta c = M \gamma(\beta + d\beta) (\beta + d\beta) c + \lambda dm \gamma(\beta)^2 \beta c + \kappa dm \gamma(\beta) \beta c + (1-\alpha) dm \gamma(w) wc, \quad (3)$$

where $\gamma(x) = 1/\sqrt{1-x^2}$ and $d\beta > 0$. The conservation of energy leads to

$$M \gamma(\beta) c^2 + dm c^2 = M \gamma(\beta + d\beta) c^2 + \lambda dm \gamma(\beta)^2 c^2 + \kappa dm \gamma(\beta) c^2 + (1-\alpha) dm \gamma(w) c^2. \quad (4)$$

The collected mass is a function of the ramjet speed and is given by

$$dm = A \rho \beta c dt. \quad (5)$$

III. ACCELERATION

Taking into account equations (3)-(5), it is possible to compute the acceleration $\varphi = dv/dt$ of the ramjet, measured in the rest frame of the interstellar medium. With this aim, it is natural to define characteristic acceleration φ_* , time t_* and length x_* by the following relations

$$\varphi_* = \frac{A \rho c^2}{M}, \quad t_* = \frac{c}{\varphi_*}, \quad x_* = \frac{c^2}{\varphi_*} = ct_*. \quad (6)$$

It is also useful to introduce the following notations

$$u' = u(2-u), \quad \bar{u} = 1-u, \quad (7)$$

with $1-u' = \bar{u}^2$.

It is worth noting that ω is only 0.0071 for the most energetic known fusion reaction [1]. So, for all fusion reactions, we have $\omega' \approx 2\omega$ and $\bar{\omega} \approx 1$. If the particles collected by the ramjet are an ideal mixing of matter and antimatter, the mass reaction can be totally converted in pure energy ($\omega = 1$ and $\kappa = 0$) [6].

Using the relation

$$\gamma(\beta + d\beta) = \gamma(\beta) + \gamma(\beta)^3 \beta d\beta, \quad (8)$$

the elimination of the reduced speed w of the exhausted reaction mass by the relation $\gamma(w)^2(1-w^2) = 1$ gives a second degree equation in φ , whose physical solution is (in the following equations, the simplified notation γ means more precisely $\gamma(\beta)$)

$$\frac{\varphi}{\varphi_*} = \frac{\sqrt{\gamma^2-1}}{\gamma^3} \left(\sqrt{\gamma^2-1} + F(\gamma) - \sqrt{\gamma^2-1} \right) \quad \text{with} \quad (9)$$

$$F(\gamma) = -\lambda' \gamma^2 - 2\kappa \bar{\lambda} \gamma + \kappa^2 + \alpha'. \quad (10)$$

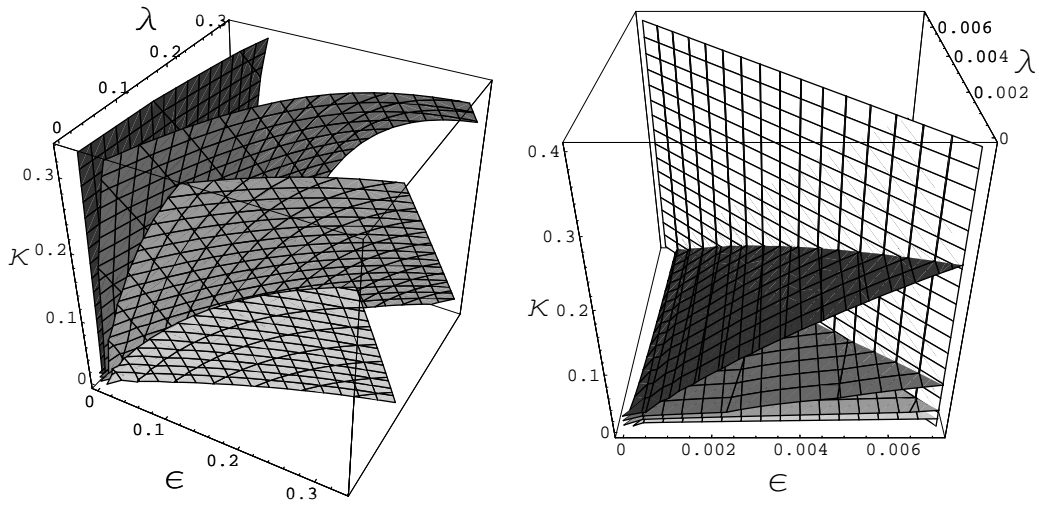


FIG. 1: Limit reduced speed β_l of the ramjet as a function of the parameters ϵ , λ and κ . Surfaces colored in dark grey to light grey correspond respectively to $\beta_l = 0.2, 0.4, 0.6, 0.8$. Left: Arbitrary values of ϵ , λ and κ . Right: Values of $\omega \leq 0.0071$ with the vertical white plane marking the constraint $\omega = 0.0071$.

This equation gives the acceleration of the ramjet as a function of its speed in the inertial frame of the interstellar medium. If $\alpha = 0$ ($\epsilon = \lambda = \kappa = 0$), the acceleration vanishes as there is no input of energy into the engine. Since $\varphi = 0$ if $\beta = 0$ ($\gamma = 1$), an initial boosting is necessary for the ramjet. This is due to the fact that the reaction mass reaches the reactor thanks to the speed of the ramjet. In theory, a very small speed is sufficient to start the ramjet. In practice, a fusion reactor could probably not operate correctly without a sufficient intake. Nevertheless, the ramjet could accelerate with an initial speed as low as 10 km/s [1]. Such a speed could be reached with usual chemical rockets or by future nuclear rockets. As expected, the acceleration also vanishes for $\beta = 1$ ($\gamma = \infty$) since no object can move at the speed of light.

But this limit speed is never reached for a non-ideal ramjet. It is clear from Eq. (9) that $\varphi = 0$ when $F(\gamma) = 0$. $F(\gamma)$ is a quadratic function in γ with two roots $-\Gamma_l$ and γ_l such that

$$F(\gamma) = \lambda'(\Gamma_l + \gamma)(\gamma_l - \gamma) \quad \text{with} \quad (11)$$

$$\gamma_l = \frac{\sqrt{\kappa^2 + \lambda'\alpha'} - \bar{\lambda}\kappa}{\lambda'} \quad \text{and} \quad \Gamma_l = \frac{\sqrt{\kappa^2 + \lambda'\alpha'} + \bar{\lambda}\kappa}{\lambda'}. \quad (12)$$

With a little algebra, it can be shown that

$$1 < \gamma_l \leq \Gamma_l, \quad (13)$$

taking into account the condition $\alpha < 1$ to obtain a non-vanishing exhausted mass. For relevant values of the speed ($\gamma \geq 1$), one can see that: $F(\gamma_l) = 0$, $F(\gamma) \geq 0$ for $\gamma \in [1, \gamma_l]$ and $F(\gamma) \leq 0$ for $\gamma \geq \gamma_l$. So, γ_l is a limit value: if $\gamma = \gamma_l$, the ramjet acceleration vanishes ($\varphi = 0$); if $\gamma < \gamma_l$, the ramjet speed increases ($\varphi > 0$); if $\gamma > \gamma_l$, the ramjet speed decreases ($\varphi < 0$). Thus, the ramjet speed tends toward the limit reduced speed $\beta_l = \sqrt{1 - \frac{1}{\gamma_l^2}}$. In the following, we will only consider the realistic case $\beta < \beta_l$ or $\gamma < \gamma_l$. Contrary to the case of an ideal ramjet, the velocity of a ramjet with radiation and/or mass losses cannot be arbitrarily close to the speed of light [6] (see Figs. 1 and 2).

Even with small values of the parameters λ and κ , the maximum speed can differ significantly from 1. This means that the slowing down of time on board the spaceship can become not large enough to allow interstellar travels in a period of time bearable for human beings. For values of $\lambda + \kappa \gg \epsilon$, the motion of the ramjet becomes non-relativistic.

In our previous papers [6, 7], all equations were given as function of the reduced speed β . We find here more convenient to present the parametric equations of motion of the ramjet as a function of its reduced speed through the kinematical factor $\gamma = \gamma(\beta)$. We will assume that, at a time $t = 0$ in the inertial frame, the position of the ramjet is $x = 0$ and its reduced speed is $\beta_0 \neq 0$. Moreover, the clocks on board the ramjet indicate a proper time $\tau = 0$.

Since

$$\frac{d\gamma}{dt} = \gamma^3 \beta \frac{d\beta}{dt} = \gamma^2 \sqrt{\gamma^2 - 1} \frac{\varphi}{c}, \quad (14)$$

the solution of Eq. (9) is given by the following integral

$$\frac{1}{t_*} \int_0^t dt = \int_{\gamma_0}^{\gamma} \frac{\gamma d\gamma}{(\gamma^2 - 1) \left(\sqrt{\gamma^2 - 1 + F(\gamma)} - \sqrt{\gamma^2 - 1} \right)}. \quad (15)$$

We give here the main steps of the procedure to solve this integral:

- To multiply the numerator and the denominator of the fraction by the quantity $(\sqrt{\gamma^2 - 1 + F(\gamma)} + \sqrt{\gamma^2 - 1})$ to obtain the function $(\gamma^2 - 1)F(\gamma)$ at the denominator;
- Using the relation (11), to write the fraction obtained as a sum of simpler fractions;
- To integrate each of these new fractions and to simplify the result.

The calculation is very heavy, but an analytical form can be found. To get a concise writing, it is useful to define some intermediate quantities:

$$\Lambda_1(z) = \bar{\lambda} - z, \quad (16)$$

$$\Lambda_2(z) = \bar{\lambda}z - \kappa, \quad (17)$$

$$U(z, \gamma) = \sqrt{(z^2 - 1)(\gamma^2 - 1)} + z\gamma - 1, \quad (18)$$

$$V(z, \gamma) = 2\sqrt{(\Lambda_2^2(z) - \bar{\alpha}^2)(\Lambda_2^2(\gamma) - \bar{\alpha}^2) + 2\bar{\lambda}\Lambda_2(z)\gamma - 2(\kappa\Lambda_2(z) + \bar{\alpha}^2)}, \quad (19)$$

$$W_{\pm}(z, \gamma) = 2\sqrt{(\Lambda_1^2(z) - \bar{\alpha}^2)(\Lambda_2^2(\gamma) - \bar{\alpha}^2) \pm 2\bar{\lambda}\Lambda_1(z)\gamma - 2(z\Lambda_1(z) + \bar{\alpha}^2)}, \quad (20)$$

$$S_{\pm}(z, \gamma, \gamma_0) = \sqrt{\Lambda_1^2(z) - \bar{\alpha}^2} \ln \frac{(\gamma_0 \mp 1)W_{\pm}(z, \gamma)}{(\gamma \mp 1)W_{\pm}(z, \gamma_0)}, \quad (21)$$

$$R(z, \gamma, \gamma_0) = \frac{1}{\sqrt{z^2 - 1}} \ln \frac{(z - \gamma_0)U(z, \gamma)}{(z - \gamma)U(z, \gamma_0)} + \frac{\sqrt{\Lambda_2^2(z) - \bar{\alpha}^2}}{z^2 - 1} \ln \frac{(z - \gamma_0)V(z, \gamma)}{(z - \gamma)V(z, \gamma_0)}. \quad (22)$$

We can then write

$$\frac{t}{t_*} = \frac{\gamma_l R(\gamma_l, \gamma, \gamma_0) + \Gamma_l R(-\Gamma_l, \gamma, \gamma_0)}{\lambda'(\gamma_l + \Gamma_l)} - \frac{S_+(\kappa, \gamma, \gamma_0)}{2\lambda'(\gamma_l - 1)(\Gamma_l + 1)} - \frac{S_-(-\kappa, \gamma, \gamma_0)}{2\lambda'(\gamma_l + 1)(\Gamma_l - 1)}. \quad (23)$$

We can see on Fig. 2, how the speed of the ramjet tends toward the limit speed as time increases.

It is interesting to look at the limit $\kappa \rightarrow 0$. In this case, we have:

$$\gamma_l \xrightarrow{\kappa=0} \sqrt{\frac{\alpha'}{\lambda'}}, \quad (24)$$

$$\beta_l \xrightarrow{\kappa=0} \sqrt{\frac{\alpha' - \lambda'}{\alpha'}}, \quad (25)$$

$$\sqrt{\Lambda_1^2(\kappa) - \bar{\alpha}^2} \xrightarrow{\kappa=0} \sqrt{\alpha' - \lambda'}, \quad (26)$$

$$\sqrt{\Lambda_2^2(\gamma_l) - \bar{\alpha}^2} \xrightarrow{\kappa=0} \sqrt{\frac{\alpha' - \lambda'}{\lambda'}}, \quad (27)$$

where $\alpha = \epsilon + \lambda$ in the r.h.s. of these relations. Moreover, Γ_l and γ_l tend toward the same limit. In formula (23), one can see that, in the limit $\kappa \rightarrow 0$, the coefficient of the functions R tends toward $\frac{\gamma_l}{2\alpha'\beta_l}$ and the coefficient of the functions S_{\pm} tends toward $\frac{1}{2\sqrt{\alpha'\beta_l}}$, where parameters γ_l and β_l are taken for $\kappa = 0$. So, one can see that expressions for the function t/t_* obtained in this paper when the parameter κ vanishes and given by formula (19) in Ref. [7] are identical (the supplementary factor 1/2 can be absorbed in the ln-functions).

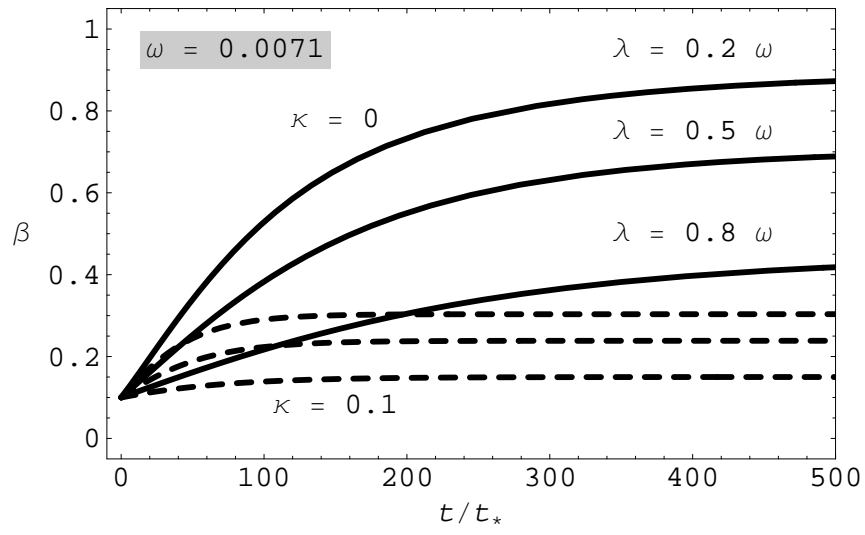


FIG. 2: Reduced speed β of the ramjet as a function of the reduced time t/t_* spent in the inertial frame of the interstellar medium, for the parameter $\omega = 0.0071$ [1] and an initial reduced speed $\beta_0 = 0.1$. The solid (dashed) curve corresponds to $\kappa = 0$ ($\kappa = 0.1$). For each value of κ , curves from bottom to top are respectively drawn for values 0.8, 0.5, 0.2 of the ratio λ/ω .

V. PROPER TIME

Using the well known relation between the time and the proper time $d\tau = dt/\gamma$ [8, 9], Eq. (15) simplifies and the proper time is given by the following integral

$$\frac{1}{t_*} \int_0^\tau d\tau = \int_{\gamma_0}^\gamma \frac{d\gamma}{(\gamma^2 - 1) \left(\sqrt{\gamma^2 - 1} + F(\gamma) - \sqrt{\gamma^2 - 1} \right)}. \quad (28)$$

An analytical solution of this integral can be found with a procedure similar to the one used for the calculation of t/t_* . Using again the notations (16)-(22), a tedious calculation gives

$$\frac{\tau}{t_*} = \frac{R(\gamma_l, \gamma, \gamma_0) - R(-\Gamma_l, \gamma, \gamma_0)}{\lambda'(\gamma_l + \Gamma_l)} - \frac{S_+(\kappa, \gamma, \gamma_0)}{2\lambda'(\gamma_l - 1)(\Gamma_l + 1)} + \frac{S_-(-\kappa, \gamma, \gamma_0)}{2\lambda'(\gamma_l + 1)(\Gamma_l - 1)}. \quad (29)$$

The link between the proper time on board the spaceship and the time spent in the inertial frame of the interstellar medium can then be computed (see Fig. 3).

Using relations (24)-(27) in formula (29), one can see that, in the limit $\kappa \rightarrow 0$, the coefficient of the functions R tends toward $\frac{1}{2\alpha'\beta_l}$ and the coefficient of the functions S_\pm tends toward $\frac{1}{2\sqrt{\alpha'}\beta_l}$, where parameters γ_l and β_l are taken for $\kappa = 0$. Again, one can see that expressions for the function τ/t_* obtained in this paper when the parameter κ vanishes and given by formula (24) in Ref. [7] are identical (the supplementary factor 1/2 can be absorbed in the ln-functions).

VI. DISTANCE

Using the relations

$$cdt = \frac{dx}{\beta} = \frac{\gamma dx}{\sqrt{\gamma^2 - 1}}, \quad (30)$$

Eq. (15) can be rewritten into the form

$$\frac{1}{x_*} \int_0^x dx = \int_{\gamma_0}^\gamma \frac{d\gamma}{\sqrt{\gamma^2 - 1} \left(\sqrt{\gamma^2 - 1} + F(\gamma) - \sqrt{\gamma^2 - 1} \right)}. \quad (31)$$

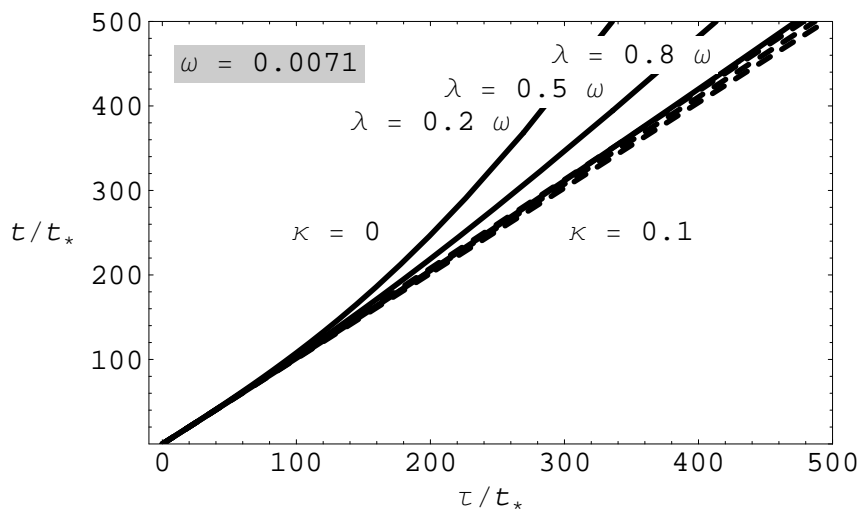


FIG. 3: Reduced time t/t_* spent in the inertial frame of the interstellar medium as a function of the reduced proper time τ/t_* on board the ramjet, for the parameter $\omega = 0.0071$ [1] and an initial reduced speed $\beta_0 = 0.1$. The solid (dashed) curve corresponds to $\kappa = 0$ ($\kappa = 0.1$). For each value of κ , curves from bottom to top are respectively drawn for values 0.8, 0.5, 0.2 of the ratio λ/ω .

Again, an analytical solution of this integral can be found with a procedure similar to the one used for the calculation of t/t_* and τ/t_* . New intermediate quantities must be defined:

$$L = \bar{\lambda} - \frac{\epsilon}{2}, \quad (32)$$

$$A = \bar{\alpha} + \frac{\epsilon}{2}, \quad (33)$$

$$Q = \frac{\bar{\lambda} \bar{\alpha}}{L A} \quad (34)$$

$$I_{\pm}(z) = \frac{\bar{\lambda} z \pm (\bar{\alpha} - \kappa)}{z \mp 1}, \quad (35)$$

$$J_{\pm}(z) = \frac{\bar{\lambda} z \pm (\bar{\alpha} + \kappa)}{z \pm 1}, \quad (36)$$

$$\nu(z) = \arcsin \sqrt{\frac{L}{J_{-}(z)}}. \quad (37)$$

We can then write

$$\frac{x}{x_*} = X(\gamma) - X(\gamma_0) \quad \text{with} \quad (38)$$

$$X(\gamma) = \frac{1}{\lambda'(\gamma_l + \Gamma_l)} \left[\ln \frac{\Gamma_l + \gamma}{\gamma_l - \gamma} + \frac{\epsilon I_{+}(\gamma_l)}{\sqrt{L A}} \Pi \left(\frac{J_{-}(\gamma_l)}{L}, \nu(\gamma), Q \right) - \frac{\epsilon I_{-}(\Gamma_l)}{\sqrt{L A}} \Pi \left(\frac{J_{+}(\Gamma_l)}{L}, \nu(\gamma), Q \right) \right], \quad (39)$$

where Π is the incomplete elliptic integral [10]. The distance travelled by the ramjet in the interstellar medium as a function of the proper time indicated by the on board clocks can then be computed (see Fig. 4).

In the limit $\kappa \rightarrow 0$, the first term of formula (39) reduces to $\frac{\gamma_l}{\alpha'} \arg \tanh \frac{\gamma}{\gamma_l}$, where parameters γ_l and β_l are taken for $\kappa = 0$. Again, one can see that expressions for the function x/x_* obtained in this paper when the parameter κ vanishes and given by formula (32) in Ref. [7] are identical.

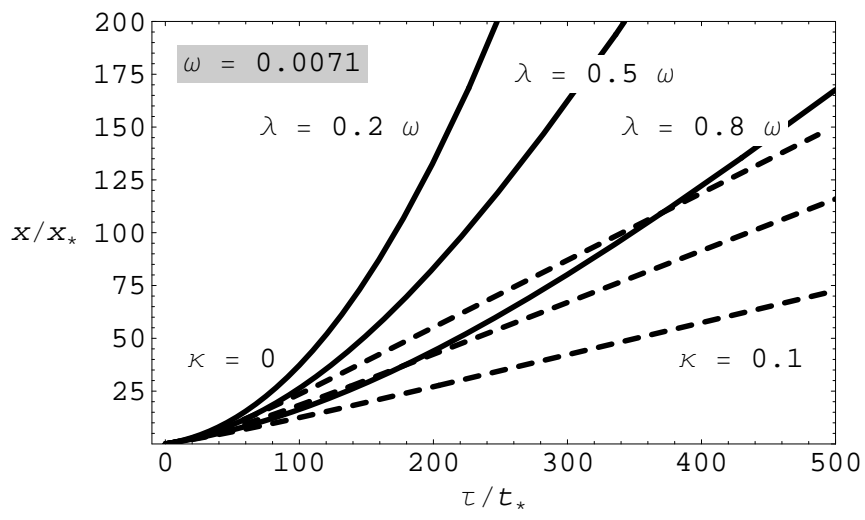


FIG. 4: Reduced distance x/x_* travelled by the ramjet in the inertial frame of the interstellar medium as a function of the reduced proper time τ/t_* on board the ramjet, for the parameter $\omega = 0.0071$ [1] and an initial reduced speed $\beta_0 = 0.1$. The solid (dashed) curve corresponds to $\kappa = 0$ ($\kappa = 0.1$). For each value of κ , curves from bottom to top are respectively drawn for values 0.8, 0.5, 0.2 of the ratio λ/ω .

VII. NON-RELATIVISTIC LIMIT

In the non-relativistic limit ($\beta \ll 1$), Eq. (9) reduces to

$$\frac{\varphi}{\varphi_*} \approx \sqrt{F(1)}\beta, \quad (40)$$

where

$$F(1) = \epsilon(2\bar{\alpha} + \epsilon). \quad (41)$$

This number is clearly positive since $\epsilon > 0$ and $\alpha < 1$. Eq. (40) can be integrated to give

$$\frac{x}{x_*} = \frac{\beta - \beta_0}{\sqrt{\epsilon(2\bar{\alpha} + \epsilon)}} \quad \text{with} \quad (42)$$

$$\beta = \beta_0 \exp\left(\sqrt{\epsilon(2\bar{\alpha} + \epsilon)} \frac{t}{t_*}\right). \quad (43)$$

These expressions, characteristic of an exponential motion [6, 7], are valid provided $t \ll -t_* \ln \beta_0 / \sqrt{\epsilon(2\bar{\alpha} + \epsilon)}$. Let us remember that τ does not differ significantly from t at non-relativistic speed.

VIII. ASYMPTOTIC MOTION

Near the limit speed, the term $F(\gamma)$ is very small with respect to the term $\gamma^2 - 1$, so we can write

$$\sqrt{\gamma^2 - 1 + F(\gamma)} - \sqrt{\gamma^2 - 1} \approx \frac{F(\gamma)}{2\sqrt{\gamma^2 - 1}}. \quad (44)$$

Equation (9) reduces then to

$$\frac{\varphi}{\varphi_*} \approx \frac{F(\gamma)}{2\gamma^3}. \quad (45)$$

It is worth noting that, in the case of an ideal ramjet ($\lambda = \kappa = 0$), $F(\gamma) = \epsilon'$ and Eq. (45) is then the equation of motion of a uniformly accelerated spaceship with a constant proper acceleration $\varphi_*\epsilon'/2$ [6].

If the ramjet is characterized by a mass $M = 1000$ t and an effective intake area $A = 10^4$ km², the characteristic acceleration is $\varphi_* \approx 1500$ m/s² for an optimistic value of the particle density, let us say $10^3/\text{cm}^3$ [1]. Within such conditions, the asymptotic proper acceleration of an ideal ramjet with $\epsilon = 0.0071$ is about 1 g. With an initial speed as low as 10 km/s, the spacecraft could then approach light velocity within a year. With a more realistic value, 1000 times smaller, for the particle density, the performance of the ramjet would be reduced by the same factor.

In the asymptotic regime, the reduced speed is very close the reduced limit speed. So we consider values of β such that $\beta_s \leq \beta < \beta_l$, with $\beta_l - \beta_s \ll \beta_l$. Using Eq. (8), the factor $\gamma_s = \gamma(\beta_s)$ can then be approximately written

$$\gamma_s = \gamma(\beta_l - (\beta_l - \beta_s)) \approx \gamma_l - \gamma_l^3 \beta_l (\beta_l - \beta_s). \quad (46)$$

So, we have

$$\beta_l - \beta_s \approx \frac{1}{\beta_l \gamma_l^3} (\gamma_l - \gamma_s). \quad (47)$$

The condition $\beta_l - \beta_s \ll \beta_l$ is then equivalent to $\gamma_l - \gamma_s \ll \gamma_l^3 \beta_l^2 < \gamma_l^3$. This inequality is relevant since γ_l has a finite value when $\lambda \neq 0$ or $\kappa \neq 0$. We assume that the ramjet is at position x_s , at time t_s , and at proper time τ_s when it has a speed β_s . Using the approximation (44), the integral for the position is given by

$$\frac{1}{x_*} \int_{x_s}^x dx \approx \int_{\gamma_s}^{\gamma} \frac{2d\gamma}{\lambda'(\Gamma_l + \gamma)(\gamma_l - \gamma)} \approx \frac{2}{\lambda'(\gamma_l + \Gamma_l)} \int_{\gamma_s}^{\gamma} \frac{d\gamma}{\gamma_l - \gamma}, \quad (48)$$

since $\gamma \approx \gamma_l$ under the integral. The corresponding solution is

$$\gamma = \gamma_l - (\gamma_l - \gamma_s) \exp \left(-\frac{\lambda'(\gamma_l + \Gamma_l)}{2x_*} (x - x_s) \right). \quad (49)$$

One can treat the time exactly in the same way. Remarking that $\gamma/\sqrt{\gamma^2 - 1} \approx 1/\beta_l$ under the integral, the solution is given by

$$\gamma = \gamma_l - (\gamma_l - \gamma_s) \exp \left(-\frac{\lambda' \beta_l (\gamma_l + \Gamma_l)}{2t_*} (t - t_s) \right). \quad (50)$$

This equation can be directly deduced from equation (49), since $x - x_s \approx \beta_l c(t - t_s)$ in this regime. Similar calculations for the proper time lead to

$$\gamma = \gamma_l - (\gamma_l - \gamma_s) \exp \left(-\frac{\lambda' \beta_l \gamma_l (\gamma_l + \Gamma_l)}{2t_*} (\tau - \tau_s) \right). \quad (51)$$

This equation can be directly deduced from equation (50), since $t - t_s \approx \gamma_l(\tau - \tau_s)$ in this regime.

Since $\beta_s \leq \beta < \beta_l$, Eq. (47) holds also for β_s replaced by β . Consequently, in Eqs. (49)-(51), quantities γ , γ_l , γ_s outside the exponentials can be replaced respectively by β , β_l , β_s . In the limit $\kappa \rightarrow 0$, it can be checked that Eqs. (49), (50), (51) tend respectively towards formulas (31), (17), (23) in Ref. [7] (note a misprint in Eq. (31): the factor β_l must be suppressed in the exponential).

IX. FUNDAMENTAL INEQUALITIES

Since $\gamma > 1$, we have

$$\tau = \int_0^\tau d\tau = \int_0^t \frac{dt}{\gamma} < \int_0^t dt = t. \quad (52)$$

Moreover, since $\beta < 1$, we have

$$x = \int_0^x dx = \int_0^t \beta c dt < \int_0^t c dt = ct. \quad (53)$$

So, from these inequalities, we can conclude that functions (23), (29) and (38) are characterized by

$$\frac{\tau}{t_*}(\gamma, \gamma_0) < \frac{t}{t_*}(\gamma, \gamma_0), \quad (54)$$

$$\frac{x}{x_*}(\gamma, \gamma_0) < \frac{t}{t_*}(\gamma, \gamma_0). \quad (55)$$

It is not evident to demonstrate these properties from the explicit form of these functions, except in the case of a perfect antimatter ramjet ($\epsilon = 1$, $\lambda = \kappa = 0$) [6]. Nevertheless, we have checked that they are fulfilled numerically in any case.

X. SUMMARY

Formulas (23), (29) and (38) form the complete set of parametric equations of motion for a Bussard ramjet with radiation and mass losses, as a function of its speed. It is then easy to compute, for instance, the distance traveled by the ramjet in the interstellar medium as a function of the proper time indicated by the on board clocks (see Fig. 4), or the link between this proper time and the time spent in the inertial frame of the interstellar medium (see Fig. 3).

With radiation and mass losses, the ramjet speed cannot be arbitrarily close to the speed of light. A limit speed, only reached asymptotically (see Figs. 1 and 2), lowers the performance of a Bussard ramjet as an interstellar spaceship or as a time machine for the exploration of the future [11].

XI. ACKNOWLEDGMENTS

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- [1] R. W. Bussard, *Astronaut. Acta* **6**, 179 (1960).
 - [2] J. F. Fishback, *Astronaut. Acta* **15**, 25 (1969).
 - [3] E. Mallove and G. Matloff, *The Starflight Handbook*, Wiley (New York, 1989).
 - [4] G. Matloff, *Deep-Space Probes*, Springer (London, 2000).
 - [5] G. Marx, *Astronaut. Acta*, **9** 131 (1963).
 - [6] C. Semay and B. Silvestre-Brac, *Eur. J. Phys.* **26**, 75 (2005).
 - [7] C. Semay and B. Silvestre-Brac, *Acta Astronaut.* **61**, 817 (2007).
 - [8] F. W. Sears and R. W. Brehme, *Introduction to the theory of relativity*, Addison-Wesley (London, 1968).
 - [9] C. Semay and B. Silvestre-Brac, *Relativité restreinte. Bases et applications*, Dunod (Paris, 2005).
 - [10] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press (New York, 1980).
 - [11] P. J. Nahin, *Time Machines*, AIP Press/Springer (New York, 1999).